



Minimal Non-Iwasawa Finite Groups

Marius Tărnăuceanu

Abstract. In this note, we describe first the structure of minimal non-Iwasawa finite groups. Then we determine the minimal non-Iwasawa finite groups which are modular. Also, we find connections between minimal non-Iwasawa finite groups and the subgroup commutativity degree, and we give an example of a family of non-nilpotent modular finite groups G_n , $n \in \mathbb{N}$, whose subgroup commutativity degree tends to 1 as n tends to infinity.

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1. Introduction

Let G be a finite group and $L(G)$ be the subgroup lattice of G . A subgroup H of G is called *permutable* if $HK = KH$, for all $K \in L(G)$, and *modular* if it is a modular element of $L(G)$. Clearly, any normal subgroup is permutable and, by Theorem 2.1.3 of [9], a permutable subgroup is always modular. If all subgroups of G are permutable then we say that G is an *Iwasawa group*, while if all subgroups of G are modular (that is, the lattice $L(G)$ is modular) then we say that G is a *modular group*. The connection between these two classes of groups is very powerful: a finite group G is an Iwasawa group if and only if it is a nilpotent modular group (see e.g. Exercise 3, page 87, [9]). Note that a complete description of the structure of finite Iwasawa groups can be obtained by using Theorems 2.4.13 and 2.4.14 of Schmidt's book.

All groups considered in this paper are finite.

Given a class of groups $\mathcal{X}$, a group G is said to be a *minimal non- $\mathcal{X}$ group*, or an *$\mathcal{X}$ -critical group*, if $G \notin \mathcal{X}$, but all proper subgroups of G belong to $\mathcal{X}$.

Many results have been obtained on minimal non- $\mathcal{X}$ groups, for various choices of $\mathcal{X}$. For instance, minimal non-abelian groups were analysed by Miller and Moreno [6], while Schmidt [8] (see also [3, 7]) studied minimal non-nilpotent groups. The latter are now known as *Schmidt groups*, and their structure is as follows: a Schmidt group G is a solvable group of order $p^m q^n$ (where p and q are different primes) with a unique Sylow p -subgroup P and a cyclic Sylow q -subgroup Q , and hence G is a semidirect product of P by Q . Moreover, we have:

- if $Q = \langle y \rangle$ then $y^q \in Z(G)$;
- $Z(G) = \Phi(G) = \Phi(P) \times \langle y^q \rangle$, $G' = P$, $P' = (G')' = \Phi(P)$;
- $|P/P'| = p^r$, where r is the order of p modulo q ;
- if P is abelian, then P is an elementary abelian p -group of order p^r and P is a minimal normal subgroup of G ;
- if P is non-abelian, then $Z(P) = P' = \Phi(P)$ and $|P/Z(P)| = p^r$.

We also recall the class of minimal non-modular p -groups, whose structure has been investigated in [4].

Our main result is the following.

Theorem 1. *A finite group is a minimal non-Iwasawa group if and only if it is either a minimal non-modular p -group or a Schmidt group $G = PQ$ with P modular.*

We observe that A_4 and $SL(2, 3)$ are examples of Schmidt groups as in Theorem 1 with $P \cong \mathbb{Z}_2^2$ abelian and $P \cong Q_8$ non-abelian, respectively.

By using Theorem 1, we are able to determine those minimal non-Iwasawa groups which are modular.

Corollary 2. *The class $\mathcal{C}$ of minimal non-Iwasawa groups which are modular consists of all Schmidt groups $G = PQ$ with P cyclic of order p .*

Note that an interesting subclass of $\mathcal{C}$ is constituted by the non-trivial semidirect products $\mathbb{Z}_3 \rtimes \mathbb{Z}_{2^n}$, $n \in \mathbb{N}^*$. Also, from the proof of Corollary 2, it will follow that:

Corollary 3. *$\mathcal{C}$ is contained in the class of minimal non-cyclic groups.*

2. Proofs of the Main Results

Proof of Theorem 1. Obviously, a minimal non-modular p -group is also a minimal non-Iwasawa group. Let $G = PQ$ be a Schmidt group with P modular. Then G is non-nilpotent, and consequently non-Iwasawa. Since all proper subgroups of G are nilpotent, it suffices to prove that they are modular. Also, we may restrict to maximal subgroups. By looking to the structure of G described in Sect. 1, we infer that these are $P \times \langle y^q \rangle$ and $\Phi(P)Q_i$, $i = 1, 2, \dots, n_q$, where Q_i , $i = 1, 2, \dots, n_q$, denote the conjugates of Q . Being a direct product of

modular groups of coprime orders, $P \times \langle y^q \rangle$ is modular. On the other hand, for each i we have

$$\Phi(P)Q_i/Z(\Phi(P)Q_i) = \Phi(P)Q_i/\Phi(P)\langle y^q \rangle \cong Q_i/\langle y^q \rangle \cong \mathbb{Z}_q,$$

implying that $\Phi(P)Q_i$ is abelian. Therefore all maximal subgroups of G are modular, as desired.

Conversely, assume that G is a minimal non-Iwasawa group. We distinguish the following two cases.

Case 1. G is nilpotent.

Then G is not modular. Let $G = \prod_{i=1}^k G_i$ be the decomposition of G as a direct product of Sylow subgroups. Then we must have $k = 1$ because all proper subgroups of G are modular. Thus G is a minimal non-modular p -group.

Case 2. G is not nilpotent.

Then G is a Schmidt group, say $G = PQ$ with P and Q as we described in Sect. 1. Since P is a proper subgroup of G , it must be modular by our assumption. This completes the proof. $\square$

Proof of Corollary 2. Let G be a group contained in $\mathcal{C}$. By Theorem 1, it follows that G is a Schmidt group of type $G = PQ$ with P modular. Since G is modular, so is $G_1 = G/Z(G)$. But G_1 is again a Schmidt group of order $p^r q$ which can be written as semidirect product of an elementary abelian p -group P_1 of order p^r by a cyclic group Q_1 of order q . Suppose that $r > 1$. Then P_1 contains a proper non-trivial subgroup, say P_2 . It is easy to see that the subgroups

$$Z(G), P_1, P_2, Q_1, \text{ and } G_1$$

form a pentagon in $L(G_1)$, a contradiction. So, $r = 1$. This leads to $|P/P'| = p$, i.e. P is abelian, and consequently $|P| = p$. $\square$

Proof of Corollary 3. Let G be a group contained in $\mathcal{C}$. From the proof of Theorem 1 it follows that all maximal subgroups of G are cyclic. Hence G is a minimal non-cyclic group. $\square$

3. Minimal Non-Iwasawa Finite Groups and Subgroup Commutativity Degrees

A notion strongly connected with Iwasawa groups is the subgroup commutativity degree of a finite group G , defined in [10] by

$$sd(G) = \frac{1}{|L(G)|^2} |\{(H, K) \in L(G)^2 \mid HK = KH\}|.$$

This measures the probability that two subgroups of G commute, or equivalently that the product of two subgroups is again a subgroup. Clearly, we have

$sd(G) = 1$ if and only if G is an Iwasawa group. $sd(G)$ has been generalized to the relative subgroup commutativity degree of a subgroup H of G (see [11]):

$$sd(H, G) = \frac{1}{|L(H)||L(G)|} |\{(H_1, G_1) \in L(H) \times L(G) \mid H_1 G_1 = G_1 H_1\}|.$$

These notions lead to two functions on $L(G)$, namely

$$f, g : L(G) \longrightarrow [0, 1], f(H) = sd(H) \text{ and } g(H) = sd(H, G), \forall H \in L(G),$$

whose study is proposed in [11]. We remark that they are constant on each conjugacy class of subgroups of G . On the other hand, we have

$$|Im f| = 1 \Leftrightarrow |Im g| = 1 \Leftrightarrow G = \text{Iwasawa group}.$$

Having in mind these results, it is an interesting problem to determine the classes $\mathcal{C}_f$ and $\mathcal{C}_g$ of finite groups G such that $|Im f| = 2$ and $|Im g| = 2$, respectively. We mention that $\mathcal{C}_g$ has been studied in [5]. Also, it is clear that the minimal non-Iwasawa groups are contained in $\mathcal{C}_f$.

Another interesting problem concerning the subgroup commutativity degree is to find some natural families of groups G_n , $n \in \mathbb{N}$, whose subgroup commutativity degree tends to a constant $a \in [0, 1]$ as n tends to infinity. For $a = 0$ many examples of such families are known (see e.g. [1, 2, 10, 12]). Since minimal non-Iwasawa groups have many commuting subgroups, we expect that they will have large subgroup commutativity degrees. Indeed, this is confirmed for groups in the class $\mathcal{C}$, as shows our following theorem.

Theorem 4. *Let p and q be two primes such that $p \equiv 1 \pmod{q}$, and G_n be a group of order pq^n contained in $\mathcal{C}$. Then*

$$\lim_{n \rightarrow \infty} sd(G_n) = 1.$$

Proof. Under the notation in the previous sections, we easily infer that $L(G_n)$ consists of G_n , of all conjugates Q_i , $i = 1, 2, \dots, n_q = p$, of Q , and of all subgroups contained in $P \times \langle y^q \rangle \cong \mathbb{Z}_{pq^{n-1}}$. Then

$$|L(G_n)| = 2n + 1 + p.$$

Also, we observe that the non-normal subgroups of G_n are Q_i , $i = 1, 2, \dots, p$. For each $H \leq G_n$, let $C(H) = \{K \in L(G) \mid HK = KH\}$. Then

$$C(H) = L(G), \quad \forall H \leq G$$

and

$$C(Q_i) = L(G) \setminus \{Q_1, \dots, Q_{i-1}, Q_{i+1}, \dots, Q_p\}, \quad \forall i = 1, 2, \dots, p.$$

One obtains

$$\begin{aligned} sd(G_n) &= \frac{1}{|L(G)|^2} \sum_{H \leq G} |C(H)| = \frac{1}{|L(G)|^2} \left(\sum_{H \leq G} |C(H)| + \sum_{H \not\leq G} |C(H)| \right) \\ &= \frac{1}{|L(G)|^2} [(|L(G)| - p)|L(G)| + p(|L(G)| - p + 1)] \\ &= 1 - \frac{p^2 - p}{(2n + 1 + p)^2}, \end{aligned}$$

which clearly tends to 1 as n tends to infinity. $\square$

4. Further Research

We end our note by indicating two natural open problems concerning the above results.

Problem 1. Determine the finite groups G containing a unique non-Iwasawa proper subgroup H .

Note that in this case H must be a minimal non-Iwasawa group, and also a characteristic maximal subgroup of G . Several examples of such groups are the quasi-dihedral group $QD_{16} = \langle x, y \mid x^8 = y^2 = 1, yxy = x^3 \rangle$ and the direct products $S_3 \times \mathbb{Z}_p$, where p is an odd prime.

Problem 2. Give a complete description of the class $\mathcal{C}_f$.

Note that $\mathcal{C}_f$ contains any direct product between a minimal non-Iwasawa group and an Iwasawa group of coprime orders. Moreover, if G is a group in $\mathcal{C}_f$, then by choosing a subgroup H of G which is minimal with the property $1 \neq sd(H) = sd(G)$, it follows that H is minimal non-Iwasawa. So, $\mathcal{C}_f$ is strongly connected to the class of minimal non-Iwasawa groups.

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Marius Tărnăuceanu
Faculty of Mathematics
“Al.I. Cuza” University
Iasi
Romania
e-mail: tarnauc@uaic.ro

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